N-62-4-3

D1-82-0184

BOEING SCIENTIFIC RESEARCH LABORATORIES

A Note on Rational Approximations

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Mathematics Research

June 1962

D1-82-0184

A NOTE ON RATIONAL APPROXIMATIONS

by

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Summer 1962

Mathematical Note No. 261

Mathematics Research Laboratory

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In an interesting report* on rational approximations for the function $x^{1-\alpha}$, Kopal suggested the following procedure. Start with the approximation

$$\Phi_0(x) = \frac{x}{1 - \alpha + \alpha x} .$$

Since the function $f(x) = x^{1-\alpha}$ satisfies a functional equation $f(x) = 1 / f(1/x)$, the function $\overline{\Phi}_0$ defined by the equation

$$\overline{\Phi}_0(x) = 1 / \Phi_0(1/x)$$

is also an approximation to $x^{1-\alpha}$. Now form the new approximation

$$\Phi_1 = \tfrac{1}{2}(\Phi_0 + \overline{\Phi}_0) ,$$

and repeat the entire process starting with Φ_1. It is clear that we may generate in this manner a sequence of functions $\Phi_0, \Phi_1, \Phi_2, \ldots$ all of which are rational. As an example of this approximation, we may take $x = 2$ and $\alpha = \frac{1}{2}$ to get, following Kopal, the following approximations for $\sqrt{2}$:

$$\Phi_0(2) = 4/3 = 1.3333333$$

$$\overline{\Phi}_0(2) = 3/2 = 1.5000000$$

$$\Phi_1(2) = 17/12 = 1.4666667$$

$$\overline{\Phi}_1(2) = 24/17 = 1.4117647$$

$$\Phi_2(2) = 577/408 = 1.4142156 .$$

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* Zdenek Kopal, "The Approximation of Fractional Powers by Rational Fractions", U. S. Army Mathematics Research Center, Report No. 119, December 1959.

The formulas that one obtains by this method are as follows:

$$\Phi_0(x) = \frac{x}{1 - \alpha + \alpha x}$$

$$\overline{\Phi}_0(x) = \alpha + (1 - \alpha)x$$

$$\Phi_1(x) = \frac{2x + \alpha(1 - \alpha)(1 - x)^2}{2(1 - \alpha) + 2\alpha x}$$

$$\overline{\Phi}_1(x) = \frac{2x(\alpha + x - \alpha x)}{2x + \alpha(1 - \alpha)(1 - x)^2}$$

etc.

Kopal expressed the conjecture that $\Phi_n(x) \to x^{1-\alpha}$ as $n \to \infty$, but reported a proof only for the case $\alpha = \frac{1}{2}$. Our analysis below will show that this case is the only one in which the conjecture is true. Kopal noted that when $\alpha = \frac{1}{2}$ the successive approximations to $\sqrt{x}$ obtained by his method were identical with those produced by Newton's method.

For the subsequent analysis we define the sequence of approximations recursively as follows:

$$\Phi_0(x) = \frac{x}{1 - \alpha + \alpha x}$$

$$\left.\begin{aligned} \overline{\Phi}_n(x) &= 1/\Phi_n(1/x) \\ \Phi_{n+1}(x) &= \tfrac{1}{2}[\Phi_n(x) + \overline{\Phi}_n(x)] \ . \end{aligned}\right\} \quad n = 0, 1, 2, \ldots$$

Fixing x and α, define $\lambda = \Phi_0(x)/\Phi_0(1/x)$. We are going to prove

that for all n,

$$\lambda = \Phi_n(x)/\Phi_n(1/x) .$$

For $n = 0$ this is true by definition. If it is true for n, then it is true for $n + 1$ since

$$\frac{\Phi_{n+1}(x)}{\Phi_{n+1}(1/x)} = \frac{\frac{1}{2}[\Phi_n(x) + \overline{\Phi}_n(x)]}{\frac{1}{2}[\Phi_n(1/x) + \overline{\Phi}_n(1/x)]} = \frac{\Phi_n(x) + 1/\Phi_n(1/x)}{\Phi_n(1/x) + 1/\Phi_n(x)}$$

$$= \frac{\lambda\,\Phi_n(1/x) + 1/\Phi_n(1/x)}{\Phi_n(1/x) + 1/\lambda\ \Phi_n(1/x)} = \lambda \quad .$$

Now define recursively $u_0 = \Phi_0(x)$ and $u_{n+1} = \frac{1}{2}(u_n + \lambda/u_n)$. We shall show that $u_n = \Phi_n(x)$ for all n. When $n = 0$ this is true by definition. If it is true for n then it is true for $n + 1$ since

$$u_{n+1} = \tfrac{1}{2}(u_n + \frac{\lambda}{u_n}) = \tfrac{1}{2}[\Phi_n(x) + \frac{\lambda}{\Phi_n(x)}]$$

$$= \tfrac{1}{2}[\Phi_n(x) + \frac{1}{\Phi_n(1/x)}] = \tfrac{1}{2}[\Phi_n(x) + \overline{\Phi}_n(x)] = \Phi_{n+1}(x) .$$

We now see that the sequence $\{u_n\}$ is the well-known Newton-Raphson iteration scheme for calculation of $\sqrt{\lambda}$. It converges if $u_0 > 0$ and $\lambda > 0$. From the definition of u_0, we see that these conditions are met whenever $x > \frac{\alpha - 1}{\alpha}$ and $x > \frac{\alpha}{\alpha - 1}$. Thus in these cases the sequence of approximations $\Phi_0(x), \Phi_1(x), \ldots$ converges to

$$\sqrt{\lambda} = \sqrt{[x(x - \alpha x + \alpha)]/(1 - \alpha + \alpha x)} .$$

In the case $\alpha = \frac{1}{2}$, $\sqrt{\lambda} = \sqrt{x}$. Note that any alteration in Φ_0 will not affect the proof of the fact that $\Phi_n(x) \to \sqrt{\lambda}$. Only the formula for λ will be changed.

<u>Remark</u> The functional equation mentioned earlier

$$f(x) = 1/\,f(1/x) \quad ,$$

which is satisfied by $f(x) = x^{\alpha}$, has for its general solution $f(x) = \exp g(\log x)$ where g is any <u>odd</u> function (i.e., $g(x) = -g(-x)$). This was pointed out to me by Professor John Green, and the proof is as follows. Put $x = e^z$ and $h(z) = f(e^z)$ so that $1 = f(x)\,f(1/x) =$ $= f(e^z)\,f(e^{-z}) = h(z)\,h(-z)$. Taking logs, $0 = \log h(z) + \log h(-z)$, and therefore $\log h(z)$ is an odd function, $g(z)$. Thus $f(x) = f(e^z) =$ $= h(z) = \exp g(z) = \exp g(\log x)$.